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Generalized SNR and Performance Index of Filters for Waveform Estimation

INTRODUCTION

For a random signal corrupted by additive statistically independent noise, it is common practice to use the SNR as a gain-invariant measure of the disturbance caused by noise. After filtering, however, the signal will be distorted and noisy; hence, the common definition of SNR is not directly applicable. The reciprocal of the normalized mean-square error, which has been used [1]–[3] is not satisfactory. We present here an adequate definition of a generalized SNR for noisy signals and discuss the SNR of optimum mean-square filters. We also indicate that the SNR performance of filters may be a misleading measure of their merit.

GENERALIZED SIGNAL-TO-NOISE RATIO

Consider a noisy signal $x(t)=s(t)+n(t)$, in which the signal $s(t)$ and the noise $n(t)$ are statistically independent stationary processes. The common SNR is $SNR = \overline{s^2(t)}/\overline{n^2(t)}$, where the bar indicates time or ensemble average. (In the following argument t is suppressed because of the stationariness of all processes involved.) For a general noisy signal y we can always write $y=cs+n_1$, in which cs is the new desired signal, n_1 is the noise, and c is a constant to be determined. The properties [4] of the SNR we would like to obtain by proper choice of c are 1) gain invariance and, 2) reduction to the common definition whenever possible. Both properties of interest are obtained by letting the "signal" cs and the "noise" n_1 be linearly independent.

$$\overline{c_1 s[y - c_1 s]} = 0 \qquad c_1 = \overline{sy}/\overline{s^2}.$$

For this constant c_1, we can write a generalized SNR:

$$SNR_G = c_1^2\overline{s^2}/\overline{[y - c_1 s]^2}$$

$$= 1\Big/\left(\frac{1}{\rho_{sy}^2} - 1\right), \qquad (1)$$

in which ρ_{sy} is the correlation coefficient.

Optimum SNR Filter

Consider y to be the result of a filtering operation of some class on a noisy signal x. We wish to find the filter, and therefore, y, which maximizes SNR_G at the output. Let $\hat{y}$ be the optimum mean-square estimate of s in the class. It is well known that the error $[s-\hat{y}]$ is uncorrelated with all outputs of the same class, i.e., $\overline{y(s-\hat{y})}=0$. Therefore, $\overline{ys}=\overline{y\hat{y}}$, in particular, $\overline{\hat{y}s}=\overline{\hat{y}^2}$. Thus,

$$\rho_{s\hat{y}} = \overline{s\hat{y}}/(\overline{\hat{y}^2}\,\overline{s^2})^{1/2} = (\overline{\hat{y}^2}/\overline{s^2})^{1/2}$$

$$\rho_{sy} = \overline{y\hat{y}}/(\overline{s^2}\,\overline{y^2})^{1/2}$$

$$= \overline{y\hat{y}}/(\overline{\hat{y}^2}\,\overline{y^2})^{1/2}\cdot(\overline{\hat{y}^2}/\overline{s^2})^{1/2} = \rho_{y\hat{y}}\rho_{s\hat{y}}$$

$$SNR_G = 1\Big/\left(\frac{1}{\rho_{y\hat{y}}^2\rho_{s\hat{y}}^2} - 1\right). \qquad (2)$$

The maximum of SNR_G is obtained for $\rho^2_{y\hat{y}}=1$, that is, $y=k\hat{y}$ (in which k is some gain factor). Therefore, the optimum mean-square filter is the optimum SNR filter, and we have

$$\max_y \{SNR_G\} = 1\Big/\left(\frac{1}{\rho^2_{s\hat{y}}} - 1\right)$$

$$= 1\Big/\left(\frac{\overline{s^2}}{\overline{\hat{y}^2}} - 1\right) = \frac{\overline{s^2}}{\overline{e^2}_{\min}} - 1.$$

in which $\overline{e^2}_{\min} = \overline{[s-\hat{y}]^2}$ is the minimum mean-square error.

The SNR performance of a filter is given by the ratio $\beta = SNR_{\text{out}}/SNR_{\text{in}}$ in which the generalized SNR is used.

PERFORMANCE INDEX FOR NOISE FILTERS

Consider the direct use of a mean-square filter in waveform estimation. Note that simple attenuation will give a mean-square error lower than the signal power, but will not furnish an improved knowledge of the signal waveform. A reasonable reference is therefore the least mean-square error achievable by attenuation, $\overline{e^2}_{\text{opt.att.}}$, and we define a performance index [5], [6]

$$\eta = \frac{\overline{e^2}_{\min}}{\overline{e^2}_{\text{opt.att.}}} = \frac{\overline{e^2}_{\min}/\overline{s^2}}{\overline{e^2}_{\text{opt.att.}}/\overline{s^2}}$$

in which $\overline{e^2}_{\min}$ is given by the optimum filter of some class and $\overline{e^2}_{\text{opt.att.}}$ is given by the optimum attenuator acting on the same input.

When the noisy signal to be filtered is an additive mixture of signal and independent noise, then the performance index η and the SNR performance β are simply related. Here, the optimum attenuator will give $\overline{e^2}/\overline{s^2}]_{\text{opt.att.}} = NSR/(1+NSR)$. Consider a graph (Fig. 1) of the normalized errors $\overline{e^2}_{\min}/\overline{s^2}$ and $\overline{e^2}_{\text{opt.att.}}/\overline{s^2}$ vs. the input noise-to-signal ratio (NSR). For a specific input NSR_{in}, the optimum filter yields a normalized error $\overline{e^2}/\overline{s^2}]_1$ (point A in Fig. 1) and the attenuator a normalized error $\overline{e^2}/\overline{s^2}]_2$.

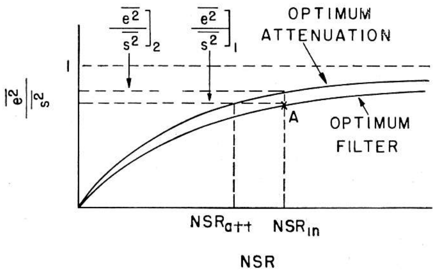

Fig. 1. Normalized error vs. NSR.

Let $NSR_{\text{att.}}$ be the input NSR for which the attenuator gives the normalized error $\overline{e^2}/\overline{s^2}]_1$. It can be shown [6] that the output NSR of the optimum filter is $NSR_{\text{out}} = NSR_{\text{att.}}$. We have, therefore,

$$\beta = NSR_{\text{in}}/NSR_{\text{att.}}\big]_{\text{constant error}}$$

$$\eta = \overline{e^2}_{\min}/\overline{e^2}_{\text{opt.att.}}\big]_{\text{constant input}}$$

Thus the two performance measures use the same system of reference, the optimum attenuator, but one compares the mean-square errors for the same input while the other compares the input NSRs for the same mean-square error.

DISCUSSION OF THE TWO MEASURES OF PERFORMANCE

The performance index indicates precisely the contribution of the filter to the desired improved knowledge of the signal waveform, while the SNR performance is an indirect measure that can be misleading. To see this, we consider, the performance of linear filters [6] as a function of NSR for additive independent noise.

It was shown [6] that if both signal and noise have spectra that do not vanish over any finite frequency bands, then

$$\overline{e^2}/\overline{s^2}]_{\text{opt.att.}} \to 1 \text{ as } NSR_{\text{in}} \to \infty,$$

$$\overline{e^2}/\overline{s^2}]_{\min} \to 1 \text{ as } NSR_{\text{in}} \to \infty$$

Therefore, the performance index, η, approaches 1 as $NSR_{\text{in}}\to\infty$, but the SNR performance, β, will in general be different from 1 as $NSR_{\text{in}}\to\infty$. Figure 1 shows the reason for this discrepancy. Although $\overline{e^2}/\overline{s^2}]_{\text{opt.filt.}}$ and $\overline{e^2}/\overline{s^2}]_{\text{opt.att.}}$ both tend asymptotically toward the same limit as $NSR_{\text{in}}\to\infty$, it takes a considerable change of NSR (from NSR_{in} to $NSR_{\text{att.}}$) for an attenuator to give the incrementally lower mean-square error of the optimum filter. It is clear that, in this case, the performance index is a more accurate measure of the performance of the filter for waveform estimation.

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